连通性以及道路连通性详细概念参见：

http://staff.ustc.edu.cn/~wangzuoq/Courses/22S-Topology/Notes/Lec17.pdf

http://staff.ustc.edu.cn/~wangzuoq/Courses/22S-Topology/Notes/Lec18.pdf

Connected 连通

$(X, \mathcal{T})$，若 $\exists\, A, B \subset X$，

$X = A \cup B$ 且 $A \cap \overline{B} = B \cap \overline{A} = \phi$

称 X disconnected

若 X 非 disconnected，称 X connected.

若 X 仅有单点集连通，称 X 完全不连通.

Fact.

$(X, \mathcal{T})$ disconnected

$\Leftrightarrow \exists\, A, B$ open, $X = A \sqcup B$

$\Leftrightarrow \exists\, A, B$ closed, $X = A \sqcup B$

$\Leftrightarrow \exists\, A \neq \phi, X$, A clopen

$\Leftrightarrow \exists\, f: X \to \{0,1\}$ cts, surj

$Y \subset X$

Y disconnected

$\Leftrightarrow \exists\, A, B \subset X$, $A \cap \overline{B}^X = B \cap \overline{A}^X = \phi$, $Y = A \cup B$

$\Leftrightarrow \exists\, A, B \subset X$ open, $A \cap Y \neq \phi$, $B \cap Y \neq \phi$, $Y \subset A \sqcup B$

$\Leftrightarrow \exists\, f: Y \to \{0,1\}$ cts surj

Prop.

1. $S \subset \mathbb{R}$ 连通 $\Leftrightarrow$ S 是区间 (Dedekind 完备性及稠密性)
2. 连通性在连续映射下保持.
3. $A_\alpha \subset X$，A_α 连通，$\bigcap_\alpha A_\alpha \neq \phi \Rightarrow \bigcup_\alpha A_\alpha$ 连通

   X, Y connected $\Rightarrow X \times Y$ connected.

   $\forall \alpha$, X_α 连通 $\Leftrightarrow \prod_\alpha X_\alpha$ 连通.

   $A \subset X$ 连通 $\Rightarrow \forall\, A \subset B \subset \overline{A}$，$B$ 连通

4. $(X, \mathcal{T})$ (T1) 且 (T4)，$|X| \geq 2 \Rightarrow |X|$ 不可数

   $(X, \mathcal{T})$ (T1) 且 (T3)，$|X| \geq 2 \Rightarrow |X|$ 不可数

Path connected.

Path: $\gamma : [0,1] \to X$ cts, $\gamma(0) = x_0$, $\gamma(1) = x_1$

$\Omega(X; x_0, x_1)$, $\Omega(X; x_0)$

$\bar{\gamma}(t) \triangleq \gamma(1-t)$;

$\gamma_1 \in \Omega(X; x_0, x_1)$, $\gamma_2 \in \Omega(X; x_1, x_2)$

定义 $\gamma_1 * \gamma_2(t) \triangleq \begin{cases} \gamma_1(2t), & 0 \le t < \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$

Path connected:

$\forall x_1, x_2 \in X$, $\Omega(X; x_1, x_2) \neq \phi$

Fact.

1. Locally Euclidean + connected $\Rightarrow$ Path connected
2. C + LPC $\Rightarrow$ PC
3. 道路连通在连续映射下保持
4. X_α P.C, $\bigcap_\alpha X_\alpha \neq \phi \Rightarrow \bigcup_\alpha X_\alpha$ P.C

   $\forall \alpha$, X_α P.C $\Leftrightarrow \prod_\alpha X_\alpha$ P.C
5. X LC $\Leftrightarrow \forall\, U \subset X$ open, U中连通分支开

   X LC, $f: X \to Y$ cts, 若 f open or closed $\Rightarrow f(X)$ LC

C:connexted; LC:locally connected; PC:path connected; LPC:locally path connected

除了 C 和 PC，其他不要求掌握，看个乐子就好

Component

$(X, \mathcal{T})$, 定义

(1) $x \sim y \Leftrightarrow \exists A$ connected, $x, y \in A$

$\Rightarrow$ 连通分支

(2) $x \underset{p}{\sim} y \Leftrightarrow \Omega(X; x, y) \neq \phi$

$\Rightarrow$ 道路连通分支.

Fact.

1. 连通分支为闭子集，道路连通分支不一定.
2. L. connected $\Rightarrow$ components are open
3. LPC $\Rightarrow$ Path components are components.